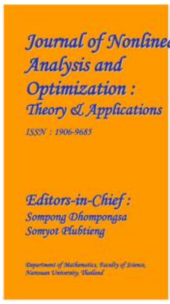

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 15, Issue. 2, No.2 : 2024
ISSN : **1906-9685**

THE STUDY OF SERIAL CHANNELS CONNECTED TO NON-SERIAL CHANNELS WITH FEEDBACK IN SERIAL CHANNELS AND RENEGING AND BALKING IN BOTH TYPES OF CHANNELS

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Abstract
A general queuing model having feedback, balking and reneging in serial queuing processes connected with non-serial queuing channels with reneging and balking in random order selection for service has been studied in the present paper. Such models are of common occurrence in the administrative setup. The mean queue length of the model when queue discipline is first come first served is obtained for the model. Numerical results have also been obtained with respect to different types of customer's behaviour. Graphs representing the mean queue length w.r.t. various parameters have been obtained.

Introduction

Various researchers including O'brien (1954), Barrer (1955) and Finch (1959) studied the problems of serial queues in steady-state with Poisson assumption. Singh (1984) studied the problem of serial queues introducing the concept of reneging. Singh and Singh (1994) worked on the network of queuing processes with impatient customers. Punam, et.al (2011) found the steady-state solution of serial queuing processes where feedback is not permitted. Satyabir, et.al (2014) obtained steady-state solution of serial queues with feedback, balking and reneging. However there may be situations where the serial queuing processes may be connected with non-serial queuing channels keeping the above observations in view, we in the present paper obtained the steady-state solutions for serial queuing processes with feedback, balking and reneging connected with non-serial queuing channels with reneging and balking in which

(i) M-serial queuing processes with feedback, balking and reneging connected with N-non-serial queuing channels with reneging and balking.
(ii) A customer may join any channel from outside and leave the system at any stage after getting service.
(iii) Feedback is permitted from each channel to its previous channel in serial channels.
(iv) The customer may balk due to long queue at each serial and non-serial service channel.
(v)The impatient customer leaves both serial and non serial service channels after wait of certain time.
(vi) The input process in serial and non-serial channels depends upon queue size and Poisson arrivals are followed.
(vii) Exponential service times are followed.
(viii) The queue discipline is random selection for service.
(ix) Waiting space is infinite.

KeyWords :
Steady-State, difference-differential, waiting space, random selection, Poisson arrivals, exponential service, feedback, balking and reneging.

1. Formulation of the Model

The system consists of the serial queues $Q_j\left(j=1,2,3,....,M\right)$ and non-serial channels $Q_{1i}(i=1,2,3,....N)$ with respective servers $S_j\left(j=1,2,3,....,M\right)$ and $S_{1i}\left(i=1,2,3,....N\right)$. Customers demanding different types of service arrive from outside the system in Poisson stream with parameters $\lambda_j\left(j=1,2,3,......,M\right)$ and $\lambda_{1i}\left(i=1,2,3,...N\right)$ at $Q_j\left(j=1,2,3,....,M\right)$ and $Q_{1i}(i=1,2,3,....N)$ but the sight of long queue at $Q_j(\mathrm{j}=1,2,3,....\mathrm{M})$ and $Q_{1i}(i=1,2,3,....N)$ may discourage the fresh customer from joining it and may decide not to enter the service channel at $Q_j(\mathrm{j}=1,2,3,....\mathrm{M})$ and $Q_{1i}(i=1,2,3,....N)$. Then the Poisson input rate at $Q_j(\mathrm{j}=1,2,3,....\mathrm{M})$ would be $\frac{\lambda_j}{n_j+1}$ where n_j is the queue size of $Q_j(\mathrm{j}=1,2,3,....\mathrm{M})$ and $\frac{\lambda_{1i}}{m_i+1}$ where m_i is the queue size of $Q_{1i}(i=1,2,3,....N)$. Further, the impatient customer joining any serial service channel $Q_j\left(j=1,2,3,....,M\right)$ and non-serial channel $Q_{1i}(i=1,2,3,....N)$ may leave the queue without getting service after wait of certain time. Here C_{in_i} and D_{jm_j} are reneging rates at which customer renege after a wait of time T_{0i} whenever there are n_i and m_j customer in the service channels Q_i and Q_{1j}.

$C_{in_i}=\dfrac{\mu_{1i}e^{-\frac{\mu_{1i}T_{0i}}{n_i}}}{1-e^{-\frac{\mu_{1i}T_{0i}}{n_i}}}\quad\left(i=1,2,3,....M\right)$ and $D_{jm_j}=\dfrac{\mu_{1j}e^{-\frac{\mu_{1j}T_{0j}}{m_j}}}{1-e^{-\frac{\mu_{1j}T_{0j}}{m_j}}}\quad\left(j=1,2,3,....N\right)$. Service time distributions for servers $S_j\left(j=1,2,3,....,M\right)$ and $S_{1i}\left(i=1,2,3,....N\right)$ are mutually independent negative exponential distribution with parameters $\mu_j\left(j=1,2,......,M\right)$ and $\mu_{1i}\left(i=1,2,3,...N\right)$ respectively. After the completion of service at S_j, the customer either leaves the system with probability p_j or joins the next channel with probability $\frac{q_j}{n_{j+1}+1}$ or join back the previous channel with probability $\frac{r_j}{n_{j-1}+1}$ such that $p_j+\frac{q_j}{n_{j+1}+1}+\frac{r_j}{n_{j-1}+1}=1\ \left(j=1,2,3,...M-1\right)$ and after the completion of service at S_M the customer either leaves the system with probability p_M or join back the previous channel with probability $\frac{r_M}{n_{M-1}+1}$ or join any queue $Q_{1i}\left(i=1,2,3,.....N\right)$ with probability $\frac{q_{Mi}}{m_i+1}\left(i=1,2,3,....N\right)$ such that $p_M+\frac{r_M}{n_{M-1}+1}+\sum_{i=1}^{N}\frac{q_{Mi}}{m_i+1}=1$. It is being mentioned here that $r_j=0$ for $j=1$ as there is no previous channel of the first channel.

The applications of such models are of common occurrence. For example, consider the administration of a particular district in a particular state at the level of district head quarter consisting of Block Development officer, Tehsildar, Sub-Divisional Magistrate, District Magistrate etc. These officers correspond to the servers of serial channels. Education Department, Health Department, Irrigation Department etc. connected with the last server of serial queue correspond to non-serial channels. The people meet the officers of the district in connection with their problems. It is also a common practice that officers call the customers (people) for hearing randomly. The senior officer may send any customer to his junior if some information regarding the customer’s problem is lacking. Further District Magistrate may send the customers to different departments such as Education, Health, Irrigation etc if there problems are related to such departments. The customer after seeing long queues before any serial and non-serial service channel may decide not to enter the queue. It generally happens that person becomes impatient after joining the queue and may leave the channel without getting service.

2. Formulation of Equations:

Define: $P\left(n_1,n_2,n_3,.....n_{M-1},n_M,m_1,m_2,m_3,.....m_{N-1},m_N;t\right)$ = the probability that at time 't' there are n_j customers (which may balk, renege or leave the system after being serviced or join the next phase or join back the previous channel) waiting before $S_j\left(j=1,2,3,....,M-1,M\right)$; m_i customers(which may balk, renege or leave the system after being serviced) waiting before the severs $S_{1i}\left(i=1,2,3,.....N\right)$.

We define the operators $T_{i\cdot}$, $T_{\cdot i}$, $T_{\cdot i,i+1\cdot}, T_{i-1\cdot,\cdot i}$ to act upon the vectors $\tilde{n}=\left(n_1,n_2,n_3,.....n_M\right)$ or $\tilde{m}=\left(m_1,m_2,m_3,.....m_N\right)$ as fallows

$$T_{i\cdot}(\tilde{n})=\left(n_1,n_2,n_3,.....,n_i-1,....n_M\right)$$
$$T_{\cdot i}(\tilde{n})=\left(n_1,n_2,n_3,.....,n_i+1,....,n_M\right)$$
$$T_{\cdot i,i+1\cdot}(\tilde{n})=\left(n_1,n_2,n_3,.....,n_i+1,n_{i+1}-1,....,n_M\right)$$
$$T_{i-1\cdot,\cdot i}(\tilde{n})=\left(n_1,n_2,n_3,.....,n_{i-1}-1,n_i+1....,n_M\right)$$

Following the procedure given by Kelly [5], we write the difference – differential equations as

$$\frac{d\,P(\tilde{n},\tilde{m};t)}{dt}=-\left[\sum_{i=1}^{M}\frac{\lambda_i}{n_i+1}+\sum_{i=1}^{M}\delta(n_i)\left(\mu_i+C_{in_i}\right)+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1}+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+D_{jm_j}\right)\right]P(\tilde{n},\tilde{m};t)$$
$$+\sum_{i=1}^{M}\frac{\lambda_i}{n_i}P\left(T_{i\cdot}(\tilde{n}),\tilde{m};t\right)+\sum_{i=1}^{M}\left(\mu_i p_i+C_{in_i+1}\right)P\left(T_{\cdot i}(\tilde{n}),\tilde{m};t\right)$$
$$+\sum_{i=1}^{M-1}\mu_i\frac{q_i}{n_{i+1}}P\left(T_{\cdot i,i+1\cdot}(\tilde{n}),\tilde{m};t\right)+\sum_{i=1}^{M}\mu_i\frac{r_i}{n_{i-1}}P\left(T_{i-1\cdot,\cdot i}(\tilde{n}),\tilde{m};t\right).$$
$$+\sum_{j=1}^{N}\mu_M\frac{q_{Mj}}{m_j}P\left(n_1,n_2,....n_M+1,T_{j\cdot}(\tilde{m});t\right)$$
$$+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P\left(\tilde{n},T_{j\cdot}(\tilde{m});t\right)+\sum_{j=1}^{N}\left(\mu_{1j}+D_{jm_j+1}\right)P\left(\tilde{n},T_{\cdot j}(\tilde{m});t\right) \qquad (2.1)$$

for $n_i\geq 0\ \left(i=1,2,3,....,M\right)$, $m_j\geq 0\left(j=1,2,3,...N\right)$;

where $\delta(x)=\begin{cases}1 & if\quad x\neq 0\\ 0 & if\quad x=0\end{cases}$

and $P(\tilde{n},\tilde{m};t)=\tilde{0}$ if any of the arguments in negative.

3. Steady–State Equations:-

We write the following Steady–state equations of the queuing model by equating the time-derivates to zero in the equation (2.1)

$$\left[\sum_{i=1}^{M}\frac{\lambda_i}{n_i+1}+\sum_{i=1}^{M}\delta(n_i)\left(\mu_i+C_{in_i}\right)+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1}+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+D_{jm_j}\right)\right]P(\tilde{n},\tilde{m})$$
$$=\sum_{i=1}^{M}\frac{\lambda_i}{n_i}P\left(T_{i\cdot}(\tilde{n}),\tilde{m}\right)+\sum_{i=1}^{M}\left(\mu_i p_i+C_{in_i+1}\right)P\left(T_{\cdot i}(\tilde{n}),\tilde{m}\right)$$
$$+\sum_{i=1}^{M-1}\mu_i\frac{q_i}{n_{i+1}}P\left(T_{\cdot i,i+1\cdot}(\tilde{n}),\tilde{m}\right)+\sum_{i=1}^{M}\mu_i\frac{r_i}{n_{i-1}}P\left(T_{i-1\cdot,\cdot i}(\tilde{n}),\tilde{m}\right).$$
$$+\sum_{j=1}^{N}\mu_M\frac{q_{Mj}}{m_j}P\left(n_1,n_2,....n_M+1,T_{j\cdot}(\tilde{m})\right)$$
$$+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P\left(\tilde{n},T_{j\cdot}(\tilde{m})\right)+\sum_{j=1}^{N}\left(\mu_{1j}+D_{jm_j+1}\right)P\left(\tilde{n},T_{\cdot j}(\tilde{m})\right) \tag{3.1}$$

for $n_i \geq 0 \ (i=1,2,3,....,M)$, $m_j \geq 0 (j=1,2,3,...N)$;

4. Steady-State Solutions:-

The solutions of the Steady-State equations (3.1) can be verified to be

$$P(\tilde{n},\tilde{m})=P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\frac{\left(\lambda_1+\frac{\mu_2 r_2\rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)}\right)^{n_1}}{\prod_{i=1}^{n_1}(\mu_1+C_{1i})}\right)\left(\frac{1}{n_2!}\frac{\left(\lambda_2+\frac{\mu_1 q_1\rho_1}{(n_1+1)\left(\mu_1+C_{1n_1+1}\right)}+\frac{\mu_3 r_3\rho_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)}\right)^{n_2}}{\prod_{i=1}^{n_2}(\mu_2+C_{2i})}\right).$$
$$\cdot\left(\frac{1}{n_3!}\frac{\left(\lambda_3+\frac{\mu_2 q_2\rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)}+\frac{\mu_4 r_4\rho_4}{(n_4+1)\left(\mu_4+C_{4n_4+1}\right)}\right)^{n_3}}{\prod_{i=1}^{n_3}(\mu_3+C_{3i})}\right)\cdots$$
$$\cdot\left(\frac{1}{n_{M-1}!}\frac{\left(\lambda_{M-1}+\frac{\mu_{M-2}q_{M-2}\rho_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)}+\frac{\mu_M r_M\rho_M}{(n_M+1)\left(\mu_M+C_{Mn_M+1}\right)}\right)^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}(\mu_{M-1}+C_{M-1i})}\right).$$
$$\cdot\left(\frac{1}{n_M!}\frac{\left(\lambda_M+\frac{\mu_{M-1}q_{M-1}\rho_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)}\right)^{n_M}}{\prod_{i=1}^{n_M}(\mu_M+C_{Mi})}\right)\left(\frac{1}{m_1!}\frac{\left(\lambda_{11}+\frac{\mu_M q_{M1}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_1}}{\prod_{j=i}^{m_1}(\mu_{11}+D_{1j})}\right).$$
$$\cdot\left(\frac{1}{m_2!}\frac{\left(\lambda_{12}+\frac{\mu_M q_{M2}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_2}}{\prod_{j=i}^{m_2}(\mu_{12}+D_{2j})}\right)\cdots\cdots\left(\frac{1}{m_N!}\frac{\left(\lambda_{1N}+\frac{\mu_M q_{MN}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_N}}{\prod_{j=i}^{m_1}(\mu_{1N}+D_{Nj})}\right) \tag{4.1}$$

$n_i \geq 0 \ (i=1,2,3,....,M)$, $m_j \geq 0 (j=1,2,3,...N)$

where

$$\rho_1 = \lambda_1 + \frac{\mu_2 r_2 \rho_2}{(n_2+1)\left(\mu_2 + C_{2n_2+1}\right)}$$

$$\rho_2 = \left(\lambda_2 + \frac{\mu_1 q_1 \rho_1}{(n_1+1)\left(\mu_1 + C_{1n_1+1}\right)} + \frac{\mu_3 r_3 \rho_3}{(n_3+1)\left(\mu_3 + C_{3n_3+1}\right)}\right)$$

$$\rho_3 = \lambda_3 + \frac{\mu_2 q_2 \rho_2}{(n_2+1)\left(\mu_2 + C_{2n_2+1}\right)} + \frac{\mu_4 r_4 \rho_4}{(n_4+1)\left(\mu_4 + C_{4n_4+1}\right)} \qquad (4.2)$$

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$$\rho_{M-1} = \lambda_{M-1} + \frac{\mu_{M-2} q_{M-2} \rho_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2} + C_{M-2n_{M-2}+1}\right)} + \frac{\mu_M r_M \rho_M}{(n_M+1)\left(\mu_M + C_{Mn_M+1}\right)}$$

$$\rho_M = \lambda_M + \frac{\mu_{M-1} q_{M-1} \rho_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1} + C_{M-1n_{M-1}+1}\right)}$$

Solving these (4.2) M-equations for ρ_M with the help of determinants, we get

$$\rho_M = \frac{\left(\begin{array}{l}
\lambda_M \Delta_{M-1} + \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} \lambda_{M-1}\Delta_{M-2} + \\
\frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} \cdot \frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)} \lambda_{M-2}\Delta_{M-3} + \dots \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
+ \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} \cdot \frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)} \dots \\
\cdot \frac{q_3\mu_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)} \lambda_3\Delta_2 \\
+ \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} \cdot \frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)} \dots \\
\cdot \frac{q_3\mu_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)} \cdot \frac{q_2\mu_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)} \lambda_2\Delta_1 \\
+ \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} \cdot \frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)} \dots \\
\cdot \frac{q_3\mu_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)} \cdot \frac{q_2\mu_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)} \cdot \frac{q_1\mu_1}{(n_1+1)\left(\mu_1+C_{1n_1+1}\right)} \lambda_1
\end{array}\right)}{\left(\Delta_{M-1} - \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} \cdot \frac{r_M \mu_M}{(n_M+1)\left(\mu_M+C_{Mn_M+1}\right)} \Delta_{M-2}\right)} \qquad (4.3)$$

Where $\Delta_M = \Delta_{M-1} - \frac{q_{M-1}\mu_{M-1}}{\left(n_{M-1}+1\right)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} . \frac{r_M\mu_M}{\left(n_M+1\right)\left(\mu_M+C_{Mn_M+1}\right)}\Delta_{M-2}$

Where

$$\Delta_1 = 1$$

$$\Delta_2 = \begin{vmatrix} 1 & -\dfrac{\frac{r_2\mu_2}{n_2+1}}{\mu_2+c_{2n_2+1}} \\ -\dfrac{\frac{q_1\mu_1}{n_1+1}}{\mu_1+c_{1n_1+1}} & 1 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} 1 & -\dfrac{\frac{r_2\mu_2}{n_2+1}}{\mu_2+c_{2n_2+1}} & 0 \\ -\dfrac{\frac{q_1\mu_1}{n_1+1}}{\mu_1+c_{1n_1+1}} & 1 & -\dfrac{\frac{r_3\mu_3}{n_3+1}}{\mu_3+c_{3n_3+1}} \\ 0 & -\dfrac{\frac{q_2\mu_2}{n_2+1}}{\mu_2+c_{2n_2+1}} & 1 \end{vmatrix}$$

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…………………………………………………………………………… (4.4)

$$\Delta_M = \begin{vmatrix} 1 & -\dfrac{\frac{r_2}{n_2+1}\mu_2}{\mu_2+C_{2n_2+1}} & 0 & 0 & - & - & - & 0 & 0 & 0 \\ -\dfrac{\frac{q_1}{n_1+1}\mu_1}{\mu_1+C_{1n_1+1}} & 1 & -\dfrac{\frac{r_3}{n_3+1}\mu_3}{\mu_3+C_{3n_3+1}} & 0 & - & - & - & 0 & 0 & 0 \\ 0 & -\dfrac{\frac{q_2}{n_2+1}\mu_2}{\mu_2+C_{2n_2+1}} & 1 & -\dfrac{\frac{r_4}{n_4+1}\mu_4}{\mu_4+C_{4n_4+1}} & - & - & - & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & - & -\dfrac{\frac{q_{M-2}}{n_{M-2}+1}\mu_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}} & 1 & -\dfrac{\frac{r_M}{n_M+1}\mu_M}{\mu_M+C_{Mn_M+1}} \\ 0 & 0 & 0 & 0 & - & - & - & 0 & -\dfrac{\frac{q_{M-1}}{n_{M-1}+1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}} & 1 \end{vmatrix}$$

Since ρ_M is obtained, so we can get ρ_{M-1} by putting the value of ρ_M in the last equation of (4.2), ρ_{M-2} by putting the values of ρ_{M-1} and ρ_M in the last but one equation of $(4,2)$. Continuing in this way, we shall obtain $\rho_{M-3}, \rho_{M-4}, ---, \rho_3, \rho_2$ and ρ_1.

Thus, we write (4.1) as under

$$p(\tilde{n},\tilde{m}) = P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\frac{(\rho_1)^{n_1}}{\prod_{i=1}^{n_1}(\mu_1 + C_{1i})}\right)\left(\frac{1}{n_2!}\frac{(\rho_2)^{n_2}}{\prod_{i=1}^{n_2}(\mu_2 + C_{2i})}\right)\left(\frac{1}{n_3!}\frac{(\rho_3)^{n_3}}{\prod_{i=1}^{n_3}(\mu_3 + C_{3i})}\right)\cdots$$

$$\cdot\left(\frac{1}{n_{M-1}!}\frac{(\rho_{M-1})^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}(\mu_{M-1} + C_{M-1i})}\right)\cdot\left(\frac{1}{n_M!}\frac{(\rho_M)^{n_M}}{\prod_{i=1}^{n_M}(\mu_M + C_{Mi})}\right).$$

$$\cdot \quad \cdot\left(\frac{1}{m_1!}\frac{\left(\lambda_{11} + \frac{\mu_M q_{M1}\rho_M}{(n_M+1)(\mu_M + C_{Mn_M+1})}\right)^{m_1}}{\prod_{j=i}^{m_1}(\mu_{11} + D_{1j})}\right)\left(\frac{1}{m_2!}\frac{\left(\lambda_{12} + \frac{\mu_M q_{M2}\rho_M}{(n_M+1)(\mu_M + C_{Mn_M+1})}\right)^{m_2}}{\prod_{j=i}^{m_2}(\mu_{12} + D_{2j})}\right).$$

$$\cdots\left(\frac{1}{m_N!}\frac{\left(\lambda_{1N} + \frac{\mu_M q_{MN}\rho_M}{(n_M+1)(\mu_M + C_{Mn_M+1})}\right)^{m_N}}{\prod_{j=i}^{m_1}(\mu_{1N} + D_{Nj})}\right) \qquad (4.5)$$

$n_i \geq 0 \ (i = 1,2,3,....,M), m_j \geq 0 (j = 1,2,3,...N)$

We obtain $P(\tilde{0},\tilde{0})$ from the normalizing conditions.

$$\sum_{\tilde{n}=\tilde{0},\tilde{m}=\tilde{0}}^{\infty} P(\tilde{n},\tilde{m}) = 1 \qquad (4.6)$$

and with the restriction that traffic intensity of each service channel of the system is less than unity. Here it is mentioned that the customers leave the system at constant rate as long as there is a line, provided that the customers are served in the order in which they arrive. Putting $C_{in_i} = C_i \quad (i = 1,2,3,...M)$ and $D_{jm_j} = D_j \quad (j = 1,2,3,....N)$ in the steady-state solution (4.1) then $\rho_i\,(i = 1,2,3,.....\mathrm{M})$ will change accordingly and the steady- state solution reduces to

$$P(\tilde{n},\tilde{\mathrm{m}}) = P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\left(\frac{\rho_1}{\mu_1 + C_1}\right)^{n_1}\right)\left(\frac{1}{n_2!}\left(\frac{\rho_2}{\mu_2 + C_2}\right)^{n_2}\right)\left(\frac{1}{n_3!}\left(\frac{\rho_3}{\mu_3 + C_3}\right)^{n_3}\right)\cdots. \qquad (4.7)$$

$$\cdot\left(\frac{1}{n_{M-1}!}\left(\frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}}\right)^{n_{M-1}}\right)\left(\frac{1}{n_M!}\left(\frac{\rho_M}{\mu_M + C_M}\right)^{n_M}\right)\cdot\left(\frac{1}{m_1!}\left(\frac{\rho_{11}}{(\mu_{11} + D_1)}\right)^{m_1}\right).$$

$$\cdot\left(\frac{1}{m_2!}\left(\frac{\rho_{12}}{(\mu_{12}+D_2)}\right)^{m_2}\right)\cdots\left(\frac{1}{m_N!}\left(\frac{\rho_{1N}}{(\mu_{1N}+D_N)}\right)^{m_N}\right)$$

where $\rho_{1j}=\lambda_{1j}+\dfrac{\mu_M q_{Mj}\rho_M}{(n_M+1)(\mu_M+C_M)}$, $\qquad j=1,2,3,\ldots N$

We obtain $P\left(\tilde{0},\tilde{0}\right)$ from (4.6) and (4.7) as

$$\left(P\left(\tilde{0},\tilde{0}\right)\right)^{-1}=\prod_{i=1}^{M}e^{\frac{\rho_i}{\mu_i+C_i}}\prod_{j=1}^{N}e^{\frac{\rho_{1j}}{\mu_{1j}+D_j}}$$

Thus $P\left(\tilde{n},\tilde{m}\right)$ is completely determined.

5. Steady-State Marginal Probabilities

Let $P\left(n_1\right)$ be the steady-state marginal probability that there are n_1 units in the queue before the first server. This is determined as

$$P\left(n_1\right)=\sum_{n_2,n_3,\ldots n_M=0}^{\infty}\ \sum_{\tilde{m}=\tilde{0}}^{\infty}P\left(\tilde{n},\tilde{m}\right)$$

$$=\sum_{n_2,n_3,\ldots n_M=0}^{\infty}\ \sum_{\tilde{m}=\tilde{0}}^{\infty}P\left(\tilde{0},\tilde{0}\right)\left(\frac{1}{n_1!}\left(\frac{\rho_1}{\mu_1+C_1}\right)^{n_1}\right)\left(\frac{1}{n_2!}\left(\frac{\rho_2}{\mu_2+C_2}\right)^{n_2}\right)\left(\frac{1}{n_3!}\left(\frac{\rho_3}{\mu_3+C_3}\right)^{n_3}\right)\cdots\left(\frac{1}{n_{M-1}!}\left(\frac{\rho_{M-1}}{\mu_{M-1}+C_{M-1}}\right)^{n_{M-1}}\right).$$

$$\cdot\left(\frac{1}{n_M!}\left(\frac{\rho_M}{\mu_M+C_M}\right)^{n_M}\right).\left(\frac{1}{m_1!}\left(\frac{\rho_{11}}{\mu_{11}+D_1}\right)^{m_1}\right)\left(\frac{1}{m_2!}\left(\frac{\rho_{12}}{\mu_{12}+D_2}\right)^{m_2}\right)\cdots\left(\frac{1}{m_N!}\left(\frac{\rho_{1N}}{\mu_{1N}+D_N}\right)^{m_N}\right)$$

Thus $P\left(n_1\right)=\dfrac{1}{n_1!}\left(\dfrac{\rho_1}{\mu_1+C_1}\right)^{n_1}e^{-(\frac{\rho_1}{\mu_1+C_1})}\qquad n_1>0$

Similarly

$$P\left(n_2\right)=\frac{1}{n_2!}\left(\frac{\rho_2}{\mu_2+C_2}\right)^{n_2}e^{-(\frac{\rho_2}{\mu_2+C_2})}\qquad n_2>0$$

$$P\left(n_M\right)=\frac{1}{n_M!}\left(\frac{\rho_M}{\mu_M+C_M}\right)^{n_M}e^{-(\frac{\rho_M}{\mu_M+C_M})}\qquad n_M>0$$

Further let $P\left(m_1\right),P\left(m_2\right),P\left(m_3\right),\ldots,P\left(m_N\right)$ be the steady-state marginal probabilities that there are $m_1,m_2,m_3,\ldots,m_N$ customers waiting before server $S_{1i}(i=1,2,3,\ldots,N)$ respectively.

$$P(m_1)=\sum_{\tilde{n}=0}^{\infty}\sum_{m_2,m_3,\dots m_N=0}^{\infty}P(\tilde{n},\tilde{m})$$

$$=\sum_{\tilde{n}=0}^{\infty}\sum_{m_2,m_3,\dots m_N=0}^{\infty}P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\left(\frac{\rho_1}{\mu_1+C_1}\right)^{n_1}\right)\left(\frac{1}{n_2!}\left(\frac{\rho_2}{\mu_2+C_2}\right)^{n_2}\right)\left(\frac{1}{n_3!}\left(\frac{\rho_3}{\mu_3+C_3}\right)^{n_3}\right)\dots\left(\frac{1}{n_{M-1}!}\left(\frac{\rho_{M-1}}{\mu_{M-1}+C_{M-1}}\right)^{n_{M-1}}\right)$$

$$\cdot\left(\frac{1}{n_M!}\left(\frac{\rho_M}{\mu_M+C_M}\right)^{n_M}\right)\cdot\left(\frac{1}{m_1!}\left(\frac{\rho_{11}}{\mu_{11}+D_1}\right)^{m_1}\right)\left(\frac{1}{m_2!}\left(\frac{\rho_{12}}{\mu_{12}+D_2}\right)^{m_2}\right)\dots\left(\frac{1}{m_N!}\left(\frac{\rho_{1N}}{\mu_{1N}+D_N}\right)^{m_N}\right)$$

$$=\left(\frac{1}{m_1!}\left(\frac{\rho_{11}}{\mu_{11}+D_1}\right)^{m_1}\right)e^{-(\frac{\rho_{11}}{\mu_{11}+D_1})} \qquad m_1>0$$

Similarly

$$P(m_2)=\left(\frac{1}{m_2!}\left(\frac{\rho_{12}}{\mu_{12}+D_2}\right)^{m_2}\right)e^{-(\frac{\rho_{12}}{\mu_{12}+D_2})} \qquad m_2>0$$

....

$$P(m_N)=\left(\frac{1}{m_N!}\left(\frac{\rho_{1N}}{\mu_{1N}+D_N}\right)^{m_N}\right)e^{-(\frac{\rho_{1N}}{\mu_{1N}+D_N})} \qquad m_N>0$$

6. Mean Queue Length

Mean queue length before the server S_1 is determined by

$$L_1=\sum_{n_1=0}^{\infty}n_1P(n_1)=\sum_{n_1=0}^{\infty}n_1\frac{1}{n_1!}\left(\frac{\rho_1}{\mu_1+C_1}\right)^{n_1}e^{-(\frac{\rho_1}{\mu_1+C_1})}$$

$$=\frac{\rho_1}{\mu_1+C_1}$$

Similarly

$$L_2=\frac{\rho_2}{\mu_2+C_2}$$

$$L_M=\frac{\rho_M}{\mu_M+C_M}$$

Mean queue length before the server S_{11} is determined as

$$L_{11}=\frac{\rho_{11}}{\mu_{11}+D_1}$$

Similarly

$$L_{1j}=\frac{\rho_{1j}}{\mu_{1j}+D_j} \qquad j=2,3,\dots,N$$

Hence mean queue length of the system is

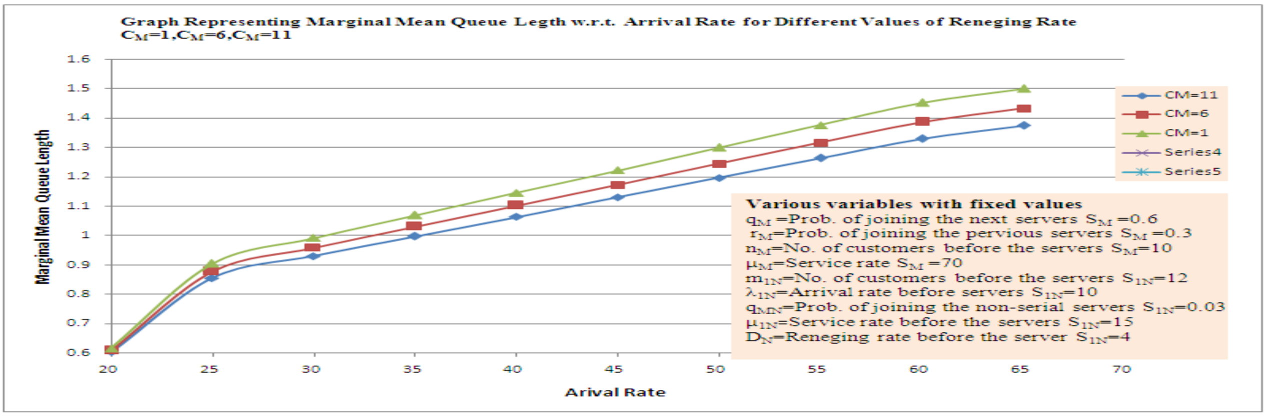
$$L=\sum_{k=1}^{M}L_k+\sum_{j=1}^{N}L_{1j}$$

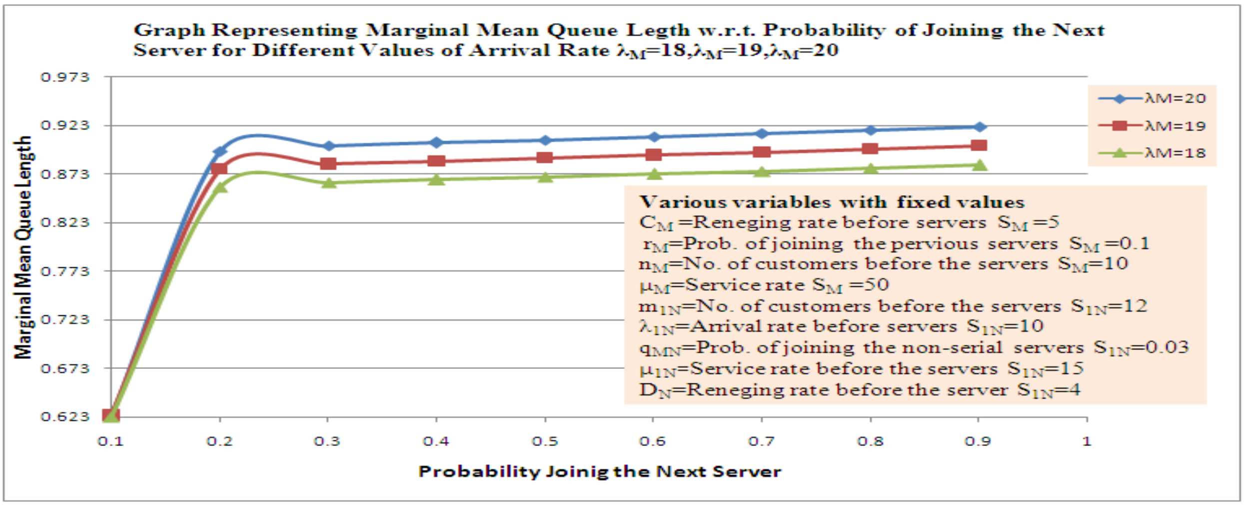
7. General Numerical Solutions of Marginal Mean Queue Length of Model

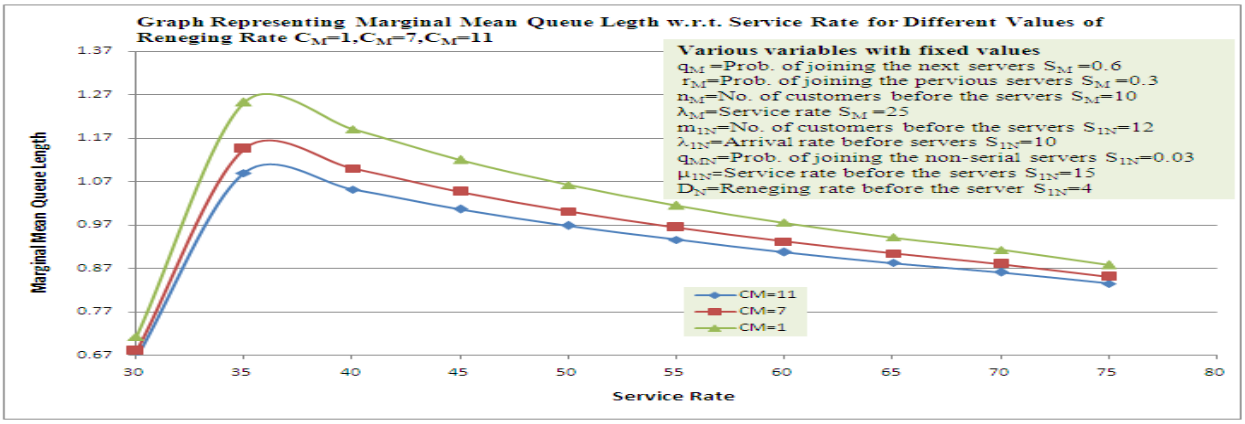
Servers in Series S_M	Arrival rate λ_M before server S_M	Reneging rate C_M before server S_M	Service rate μ_M before server S_M	Prob. of joining the next server $q_M/(N_M+1)$	Prob. of joining the previous server $r_M/(N_M+1)$	Marginal mean queue length before the server S_M $L_M = \rho_M/(\mu_M+C_M)$	Non-Serial Servers S_{1N}	Arrival rate λ_{1N} before servers S_{1N}	Prob. of joining the non-serial servers q_{MN}	Service rate μ_{1N} before servers S_{1N}	Um.Roh$_M$. $q_{MN}/(Nm+1)$ (Um+Cm)	Reneging rate before server $S_{1N}=D_N$	ρ_{1N}	Mean Queue Length before the servers $S_{1N} = \rho_{1N}/(\mu_{1N}+D_N)$	Sum of Marginal Mean Queue Lengths of Serial and Non-Serial Servers
1	12	1	14	0.0625	0	0.359074595	1	16	0.03	18	0.03726333	2	16.037263	0.801863167	1.160937762
2	14	2	15	0.04	0.03	0.86299518	2	15	0.04	16	0.04968444	3	15.049684	0.792088655	1.655083835
3	12	3	13	0.0166667	0.0333333	0.82321797	3	13	0.06	15	0.07452666	1	13.074527	0.817157916	1.640375886
4	15	1	16	0.0363636	0.0454545	0.898824311	4	14	0.07	16	0.08694777	1	14.086948	0.828643986	1.727468297
5	16	3	17	0.05	0.0071429	0.837112618	5	15	0.09	17	0.111789991	3	15.11179	0.7555895	1.592702118
6	18	2	19	0.0375	0.0125	0.923368579	6	17	0.08	19	0.099368881	4	17.099369	0.743450821	1.6668194
7	17	4	18	0.0230769	0.0461538	0.81754886	7	12	0.01	15	0.01242111	2	12.012421	0.706613006	1.524161866
8	19	1	20	0.0294118	0.0176471	0.929828612	8	14	0.07	16	0.08694777	1	14.086948	0.828643986	1.758472598
9	21	2	22	0.0727273	0.0090909	0.934018155	9	19	0.02	21	0.02484222	4	19.024842	0.760993689	1.695011843
10	23	4	25	……	0.0411765	0.844635484	10	20	0.05	22	0.06210555	5	20.062106	0.743040946	1.587676431

Mean queue length of the system = 16.00871004

8. Graphs Representing Marginal Mean Queue Length w.r.t. Various Parameters







References:

[1] Barrer, D.Y. (1955) : "A waiting line problem characterized by impatient customers and indifferent clerk". Journals of Operations Research Society of America. Vol. 3: 360

[2] Finch, P.D (1959) : "Cyclic queues". Opns. Res. Quart, Vol. 9, No. 1.

[3] Kelly, F.P (1979) : "Reversibility and stochastic networks". Wiley, New York.

[4] O'Brien, G.G. (1954) : "The solution of some queuing problems". J.Soc. Ind. Appl. Math. Vol. 2, 132-142.

[5] Punam; Singh.M; and Ashok (2011): "The Steady-State solution of serial queuing processes with reneging due to long queue and urgent message". Applied science periodical Vol. XII, No. 1

[6] Singh.M. (1984): "Steady-state behaviour of serial queuing processes with impatient customers". Math. operations forsch, U.statist. Ser. Statist.Vol. 15.No 2, 289-298

[7] Singh M. and Umed Singh(1994) : "Network of Serial and Non-serial Queuing Processes with Impatient Customers,"JISSOR,1994, Vol. 15, (1-4), 81-96.

[8] Singh. S., Taneja. G. & Singh. M. (2014): "The Steady-State Solution of Serial Channel with Feedback, Balking and Reneging", International Journal of Mathematical Sciences, ISSN:2051-5995, Vol.34, Issue.2, 1621-1628.

[9] Singh, S , Taneja, G & Singh, M (2016).Study of some serial and non-serial queuing processes with various types of customers' behaviour . Ph.D Thesis.